Topological derivatives for shape and parameter reconstruction

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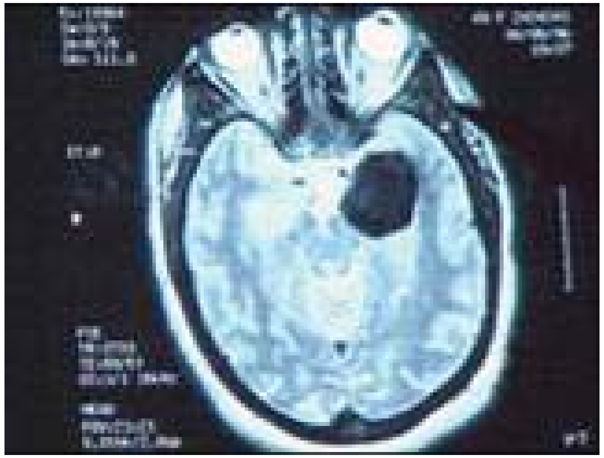
Outline

1. Inverse scattering problems
2. Topological derivative methods
   - TD for shape reconstruction
   - TD for shapes and parameters
3. Conclusions
   - Other problems
   - Conclusions

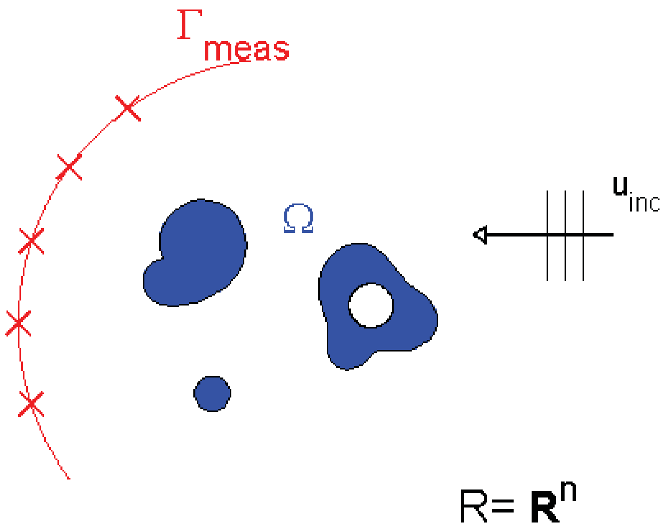
Description of the problem

Medium $\mathcal{R}$ with obstacles Ω: How many? how big? where? physical properties in Ω?



Some applications

- Medicine (tumors, fracture)
- Geophysics (oil, gas)
- Materials (damage, cracks)

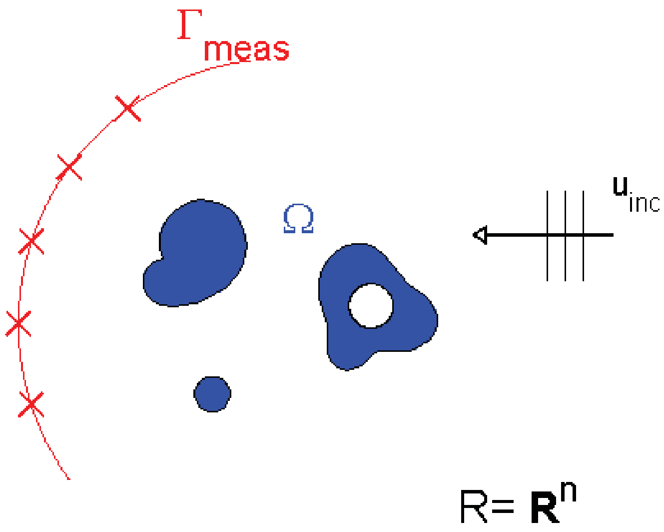


Scattering problem

An incident wave u_{inc} interacts with a medium $\mathcal{R}$ containing objects Ω.

Forward (direct) problem

- The shape, size, location and physical properties of the objects are known
- Compute the response of the system at the detectors "×"
- A well–posed problem: it has a unique solution that depends continuously on the data



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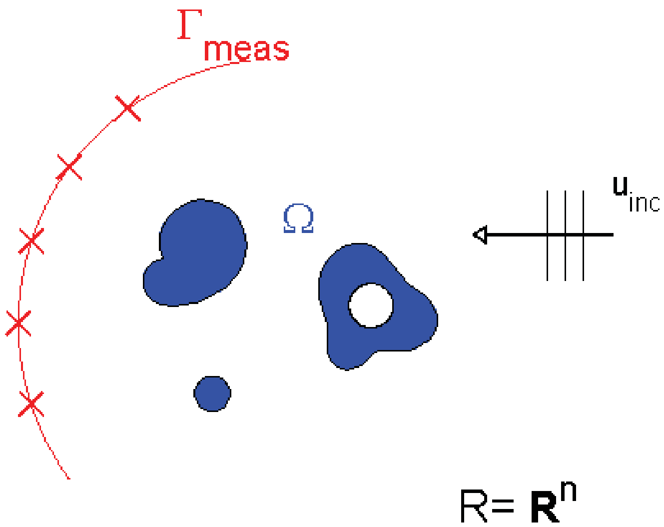

Scattering problem

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Inverse problem

- Measurements u_{meas} are taken at the receptors
- Find the scatters Ω and the interior parameters s.t.

$$u = u_{meas} \quad \text{on } \Gamma_{meas}, \quad \text{u= sol. forward problem}$$

- An ill–posed problem: it may not have a solution and if it has one, it may not depend continuously on the data

Model problem

We assume that

- We generate acustic waves
- Incident waves are time–harmonic

$$U_{inc}(\mathbf{x}, t) = \mathrm{Re}[e^{-i\omega t} u_{inc}(\mathbf{x})],$$

- u_{inc} is a planar wave in the direction $\mathbf{d}$,

$$u_{inc}(\mathbf{x}) = e^{ik\mathbf{x}\cdot\mathbf{d}}$$

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A simple forward problem

Ω is a penetrable known obstacle. The incident field generates a scattered wave u_{sc} in $\mathbb{R}^n \setminus \Omega$ and a transmitted wave u_{tr} in Ω. The total field

$$u = u_{inc} + u_{sc} \text{ in } \mathbb{R}^n \setminus \Omega \quad \text{and} \quad u = u_{tr} \text{ in } \Omega$$

solves

$$\begin{cases} \Delta u + k_e^2 u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ \Delta u + k_i^2 u = 0 & \text{in } \Omega \\ u^- = u^+, \quad \partial_n u^- = \partial_n u^+ & \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) = 0 \end{cases}$$

where $k_e, k_i > 0$ are known

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Constrained optimization

Original problem (we assume that k_i is known)

Find Ω such that

$$u = u_{meas} \quad \text{on } \Gamma_{meas}$$

A weaker formulation

Find Ω minimizing

$$J(\Omega) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

for u solving the forward problem with objects Ω

- The domain Ω is the variable
- The Helmholtz transmission problem is the constraint

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Some alternatives

Modified gradient methods: differ on how an initial guess is deformed from one iteration to the next in such a way that the cost functional decreases

- Classical deformations following a vector field
  - Problem: The number of scatterers has to be known from the beginning
  - Kirsch 1993, Hettlich 1995, Potthast 1996
- Level set based deformations allow changes in topology
  - Problem: Slow evolution. Initial guess?
  - Santosa 1996, Dorn 2005
- Topological derivatives
  - Provide good initial guesses
  - Fast and allow topological changes

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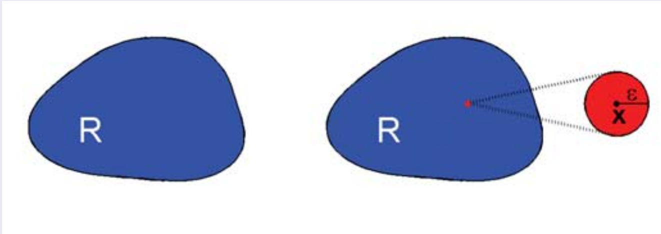
Definition of Topological Derivative (Sokowloski–Zochowski '99)

The TD of a shape functional $J(\mathcal{R})$ at a point $\mathbf{x} \in \mathcal{R}$ is

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) - J(\mathcal{R})}{\mathrm{Vol}(B_\varepsilon(\mathbf{x}))}$$



- It is a scalar function of $\mathbf{x}$
- It measures sensitivity to removing balls around $\mathbf{x}$
- $D_T(\mathbf{x}, \mathcal{R}) << 0 \implies$ high probability of finding an object

Equivalently, for $\mathbf{x} \in \mathcal{R}$ and $h(\varepsilon) = \mathrm{Vol}(B_\varepsilon(\mathbf{x}))$

$$J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) = J(\mathcal{R}) + h(\varepsilon) D_T(\mathbf{x}, \mathcal{R}) + o(h(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

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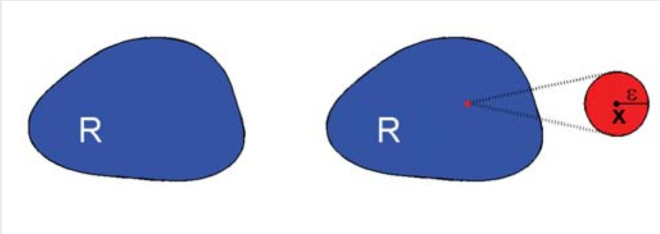

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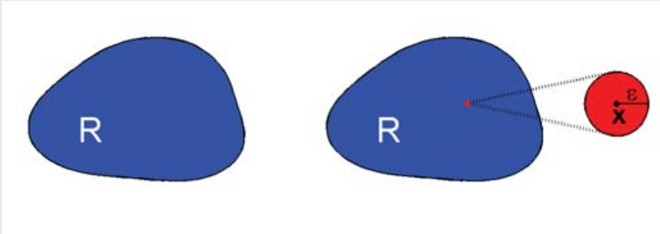

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Transmission problem: $u^- = u^+$, $\partial_n u^- = \partial_n u^+$

Case I: No a priori information on the obstacles, $\mathcal{R} = \mathbb{R}^n$, $\Omega = \emptyset$

Theorem. For any $\mathbf{x} \in \mathbb{R}^n$ the topological derivative of

$$J(\mathbb{R}^n) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

is

$$D_T(\mathbf{x}, \mathbb{R}^n) = \mathrm{Re}\left[(k_i^2 - k_e^2)\, u(\mathbf{x}) w(\mathbf{x}) \right]$$

where u and w solve forward and adjoint problems with $\Omega = \emptyset$

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Therefore, $u = u_{inc}(\mathbf{x}) = e^{ik_e \mathbf{x}\cdot\mathbf{d}}$

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Therefore, $w = \int_{\Gamma_{meas}} G_{k_e}(\mathbf{x}-\mathbf{y})(\overline{u_{meas} - u})(\mathbf{y}) dl_{\mathbf{y}}$

- The true obstacles enter in the TD through the measured data at the adjoint field
- "Free" computation: DT=Re $\left[(k_i^2 - k_e^2)\, u(\mathbf{x}) w(\mathbf{x})\right]$

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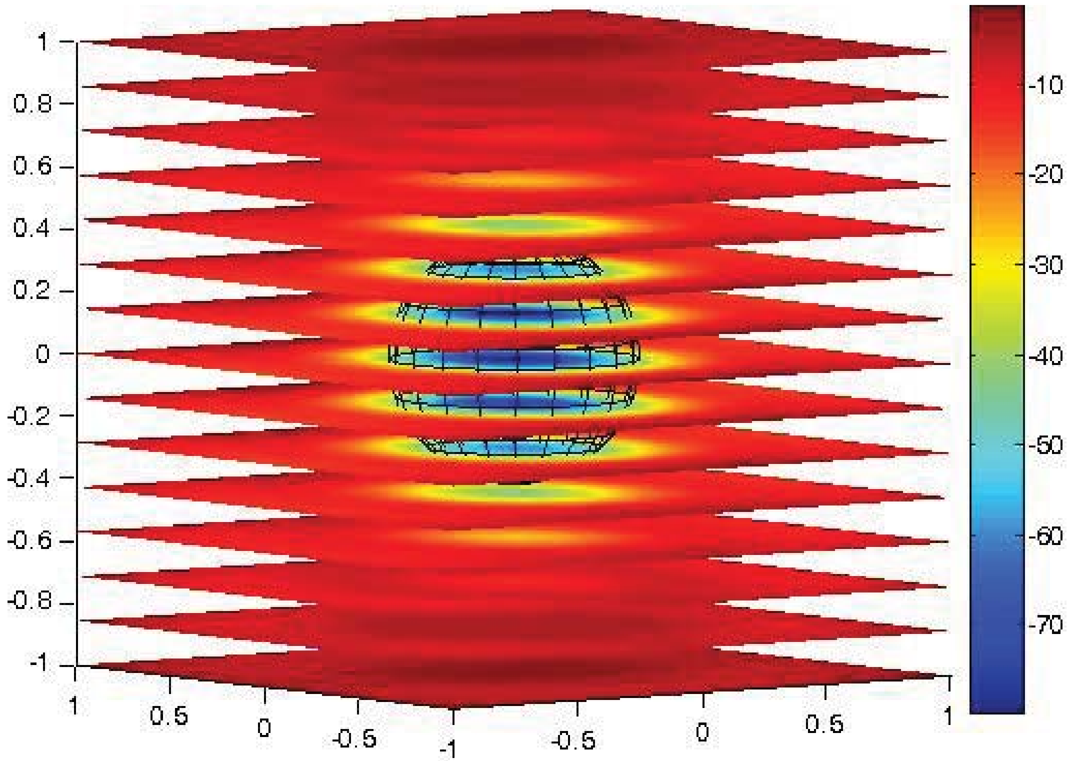

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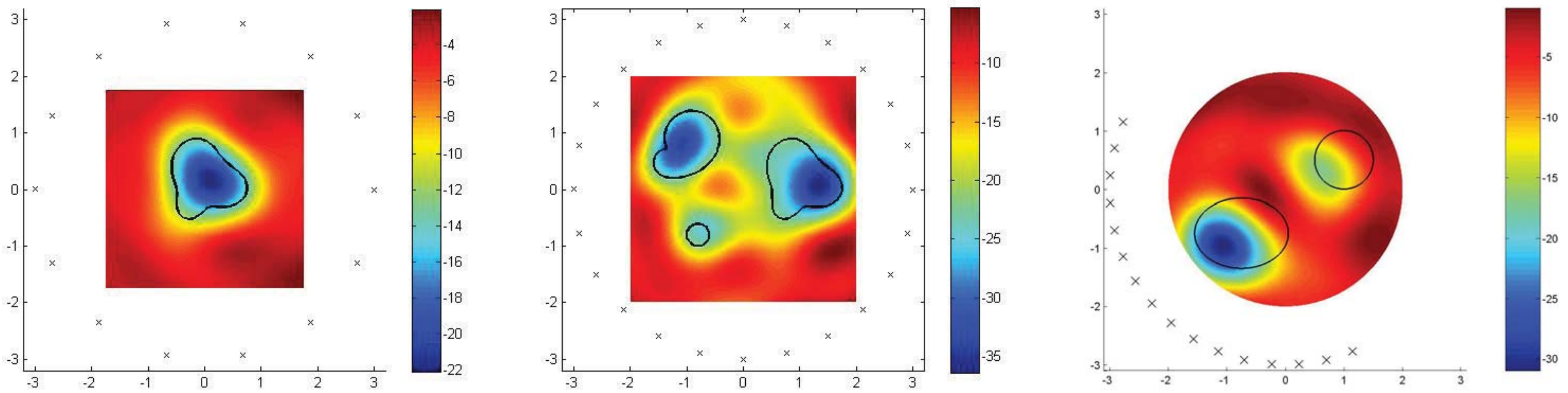

"×"= observation points, 24 incident directions in $[0, 2\pi)$, $k_e = 2$ and $k_i = 1/2$. Level of noise=1%

- Similar results when observation points are further, + observation points, + incident directions, + noise

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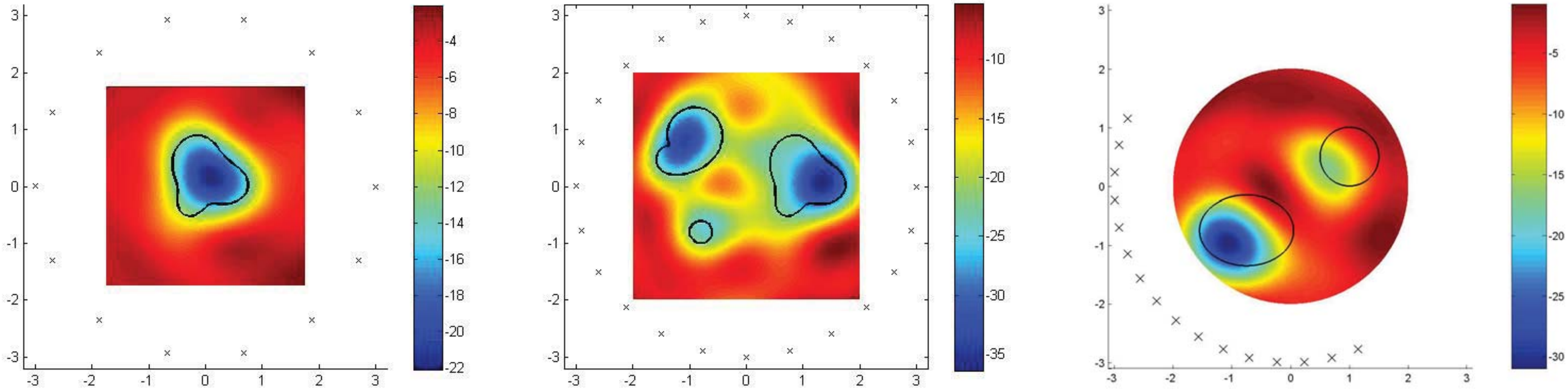

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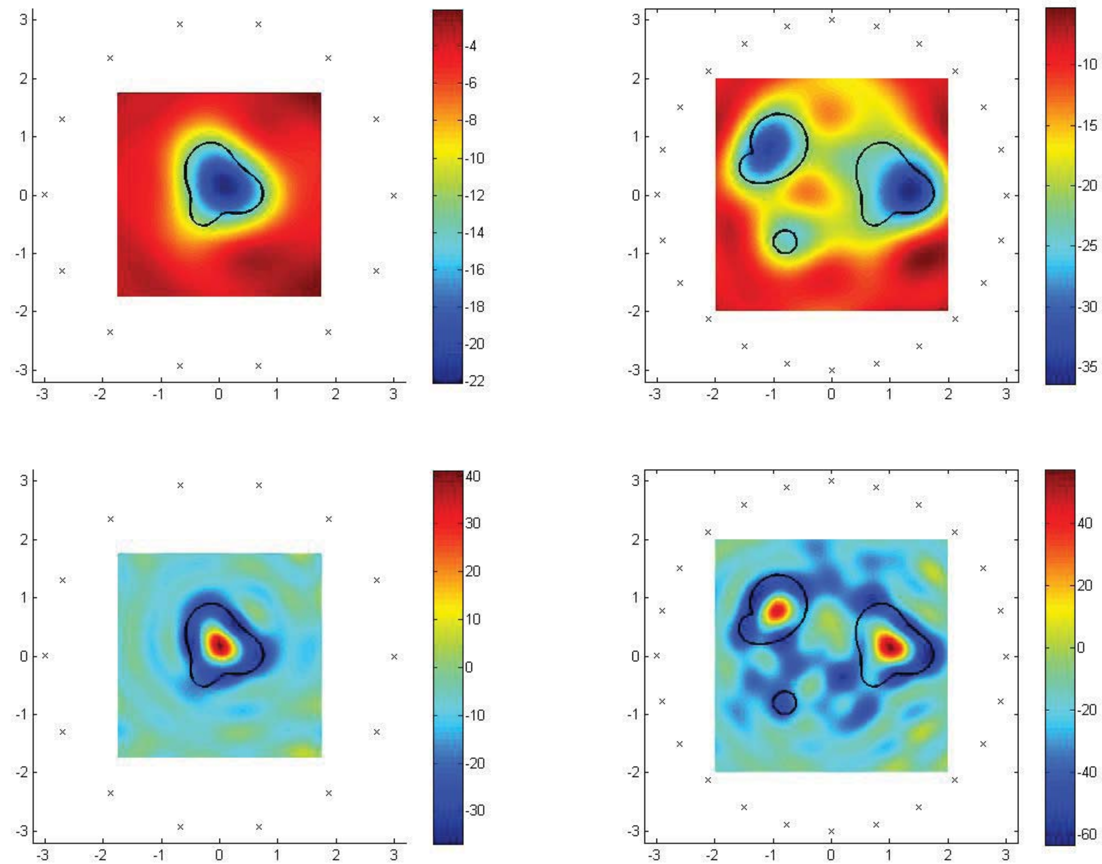

Results depend on the wave length (1 w.l.=$2\pi/k$):

1st row: $k_e = 2$ and $k_i = 1/2$

2nd row: $k_e = 4$ and $k_i = 1$

TD with an initial guess

Case II: Ω_{ap} first guess, $\mathcal{R} = \mathbb{R}^n \setminus \Omega_{ap}$, $\Omega = \Omega_{ap}$

Theorem. For any $\mathbf{x} \in \mathbb{R}^n \setminus \Omega_{ap}$ the topological derivative of

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- Forward problem with $\Omega = \Omega_{ap}$:

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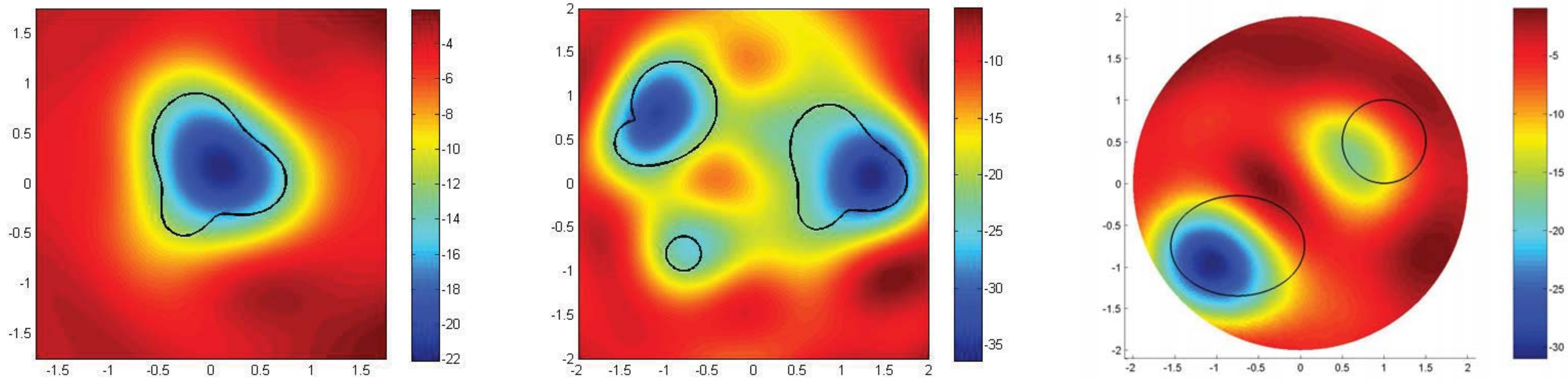
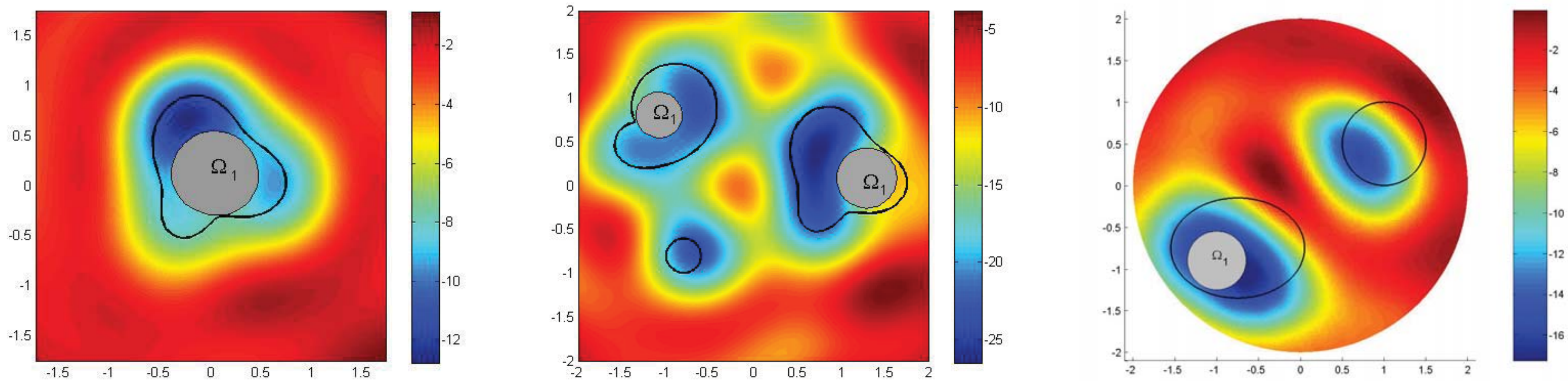
The boundary conditions influence u and w!



Same examples as before with $\Omega = \emptyset$



Initial guess Ω_1 superimposed on the TD when $\Omega = \Omega_1$

An iterative method

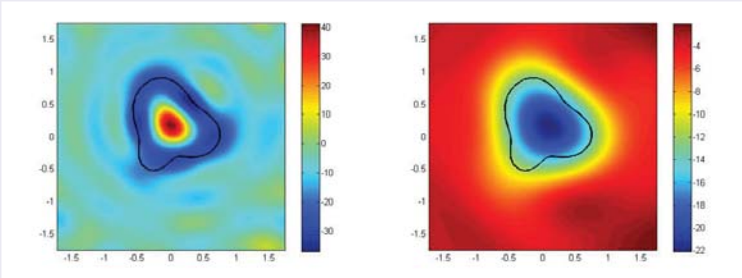
Algorithm

1. Compute the TD when $\Omega = \emptyset$
2. Take $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^n) < -C_1\}, \; C_1 > 0$
3. For j=1:jmax
   - Compute the TD in $\mathbb{R}^n \setminus \Omega_j$
   - Select $\Omega_{j+1} \supset \Omega_j$

$$\Omega_{j+1} = \Omega_j \cup \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^n \setminus \Omega_j) < -C_{j+1}\}$$

How to choose C_j?

- First step: $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2) < -C_1\}$



$$C_1 = \frac{3}{5}|\min D_T|$$

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- Iterations: $\Omega_{j+1} = \Omega_j \cup \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2 \setminus \Omega_j) < -C_{j+1}\}$

$$C_{j+1} = \frac{9}{10}|\min D_T|$$

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$$J \approx 0 \quad \text{or} \quad \Omega_j \approx \Omega_{j+1} \quad \text{or} \quad |u_\delta - u_{meas}| < 1.2\delta$$

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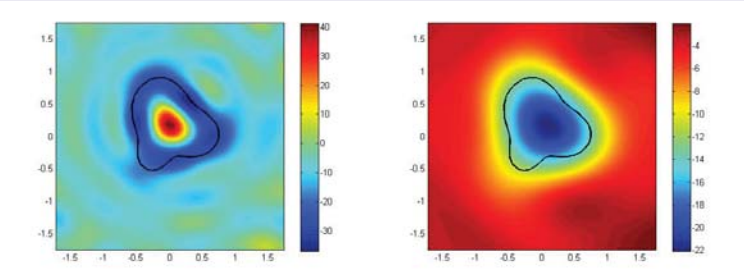

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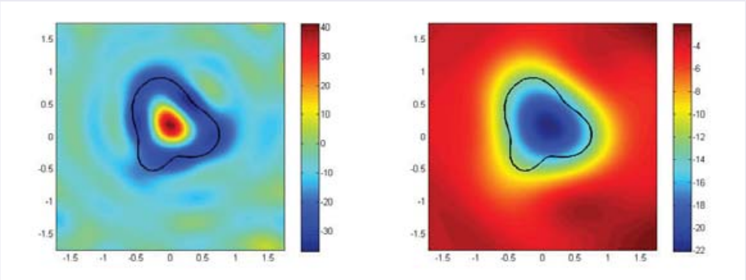
- First step: $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2) < -C_1\}$



$$C_1 = \frac{3}{5}|\min D_T|$$

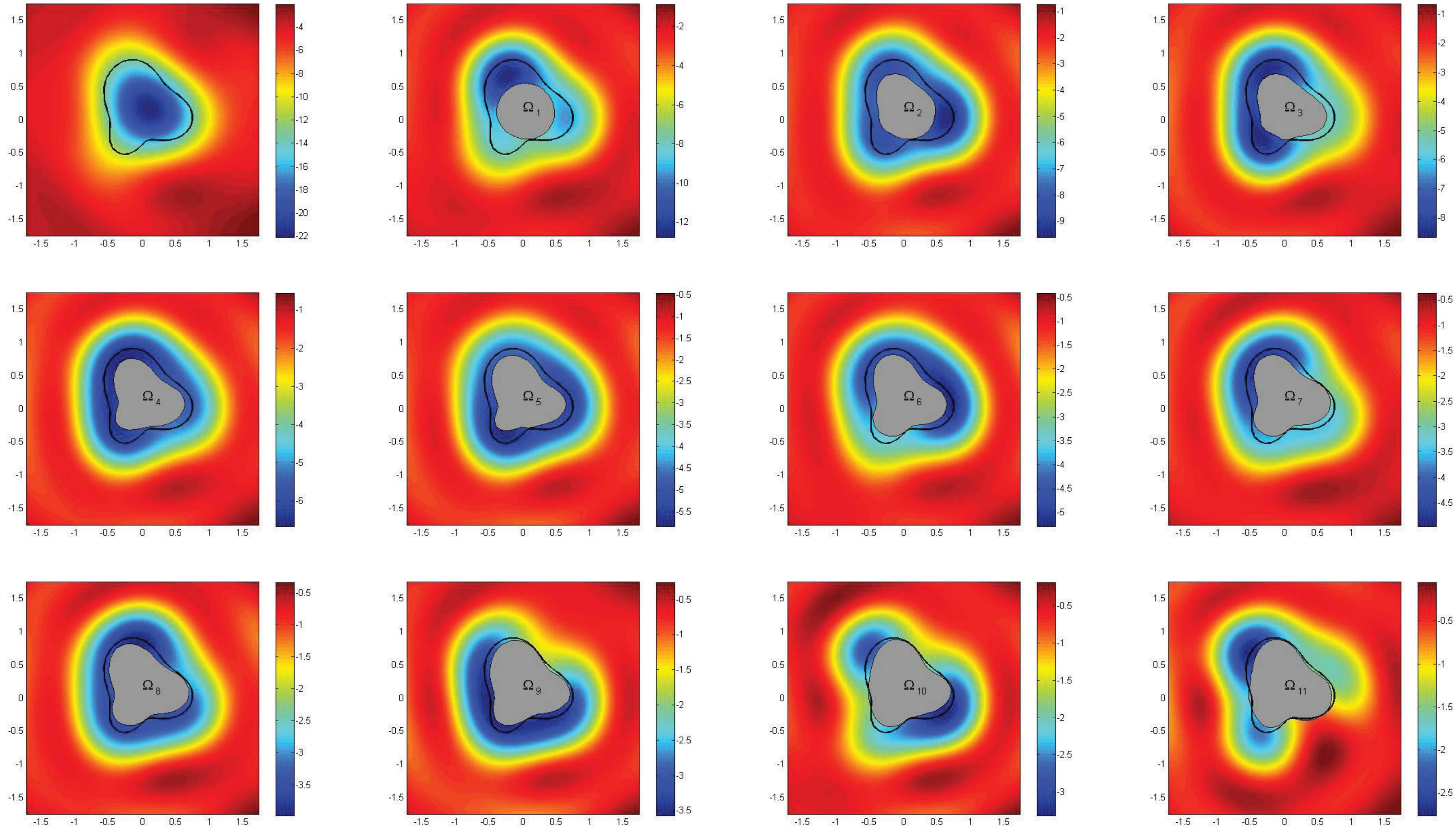
- Accept C_1 if $J_1 < J_0$
- Otherwise $C_1' < C_1$

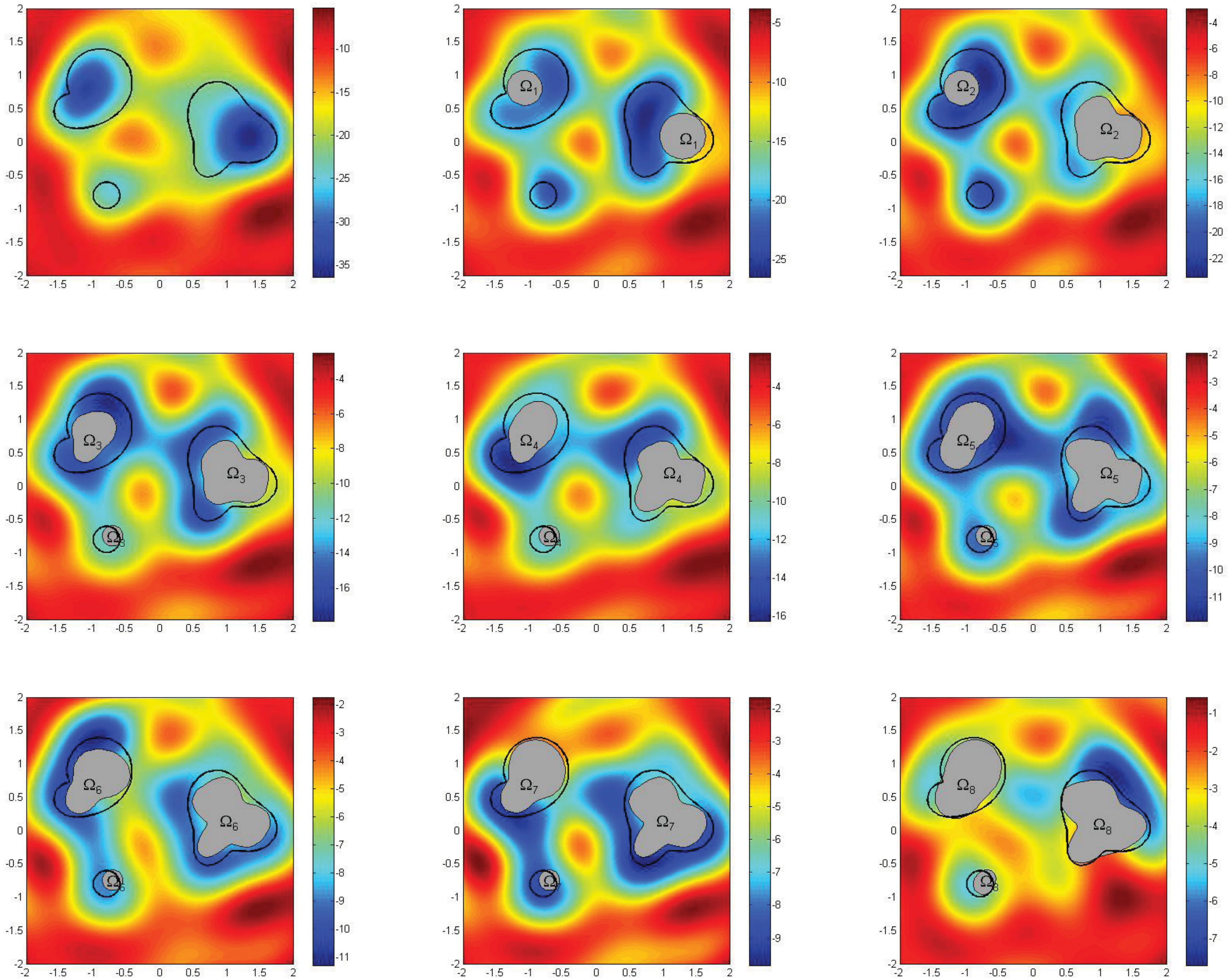
- Iterations: $\Omega_{j+1} = \Omega_j \cup \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2 \setminus \Omega_j) < -C_{j+1}\}$

$$C_{j+1} = \frac{9}{10}|\min D_T|$$

Stopping criteria?

$$J \approx 0 \quad \text{or} \quad \Omega_j \approx \Omega_{j+1} \quad \text{or} \quad |u_\delta - u_{meas}| < 1.2\delta$$


Ω 1
Ω 2
Ω 3
Ω 4
Ω 5
Ω 6
Ω 7
Ω 8
Ω 9
Ω 10
Ω 11


Ω_1
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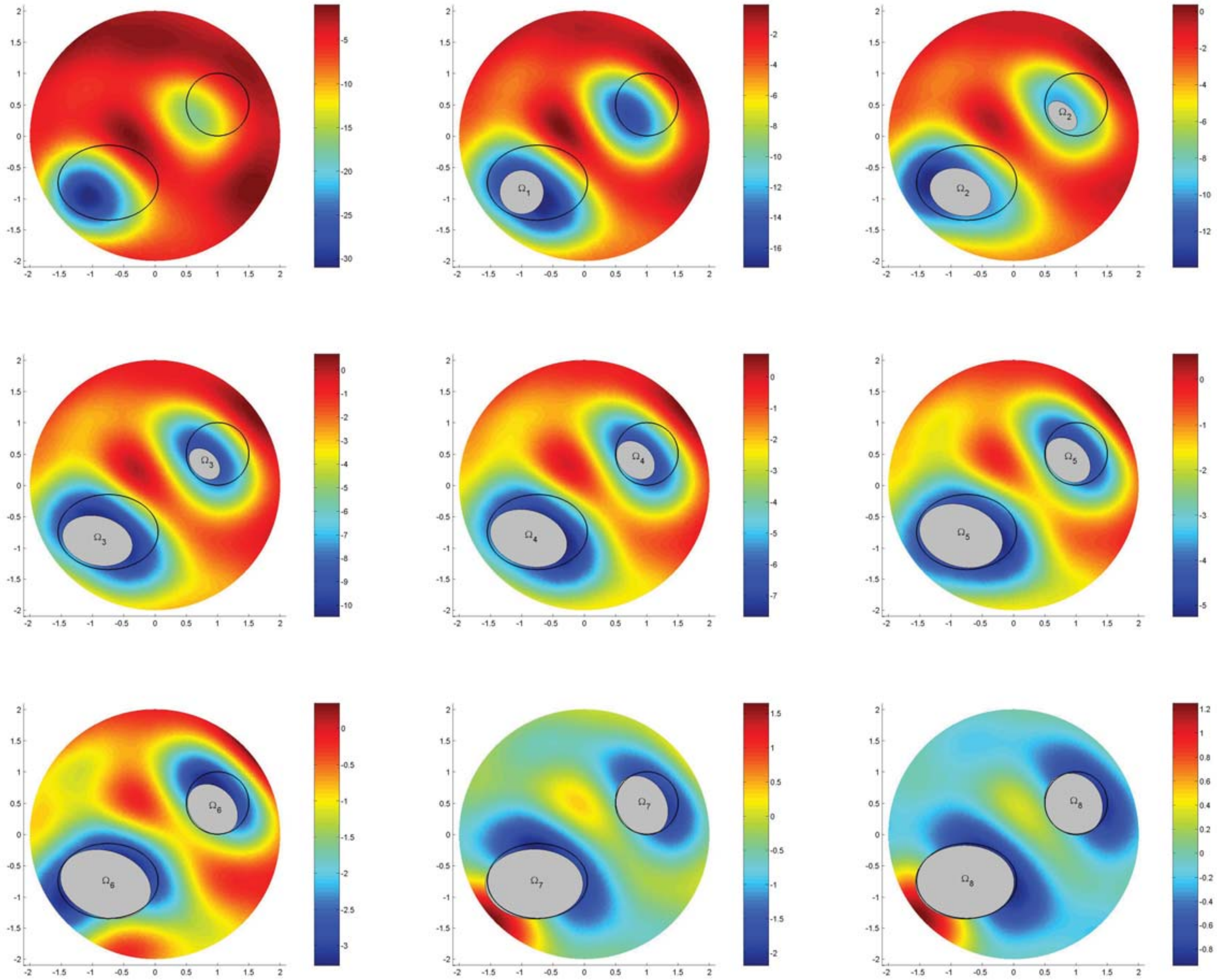
Ω1
Ω2
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Ω6
Ω7
Ω8

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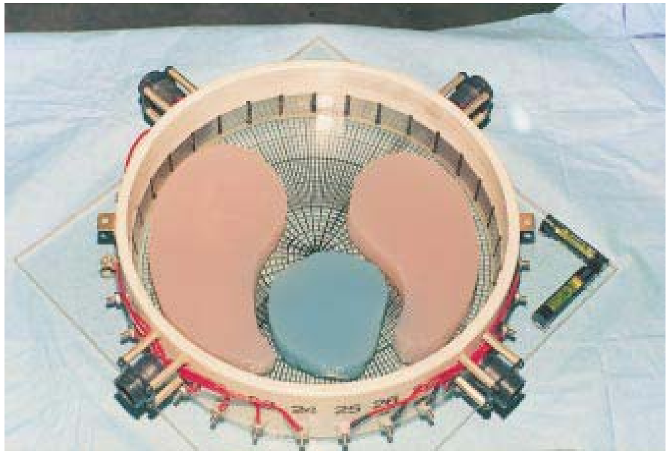

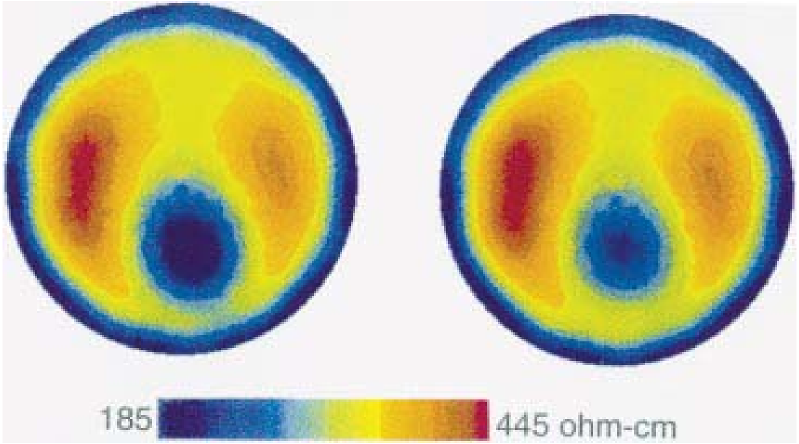

Direct problem

$$\begin{cases} \Delta u + k_e^2 u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ \Delta u + k_i^2 u = 0 & \text{in } \Omega \\ u^- = u^+, \quad \partial_n u^- = \partial_n u^+ & \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) = 0 \end{cases}$$

Inverse problem

Find Ω and k_i

Idea

- In the first computation of the TD, i.e. when $\Omega = \emptyset$, we do not need to know k_i:

$$D_T(\mathbf{x}, \mathbb{R}^2) = \mathrm{Re}\left[(k_i^2 - k_e^2)u(\mathbf{x})w(\mathbf{x})\right]$$

  where $u = u_{inc}$ and $w = \int_{\Gamma_{meas}} G_{k_e}(\mathbf{x}-\mathbf{y})(\overline{u_{meas} - u})(\mathbf{y})dl_{\mathbf{y}}$
- We compute the TD taking $k_i^0 \approx k_e$ to get an initial guess Ω_1
- In the next step, we update k_i by a gradient method
- Update Ω, update k_i

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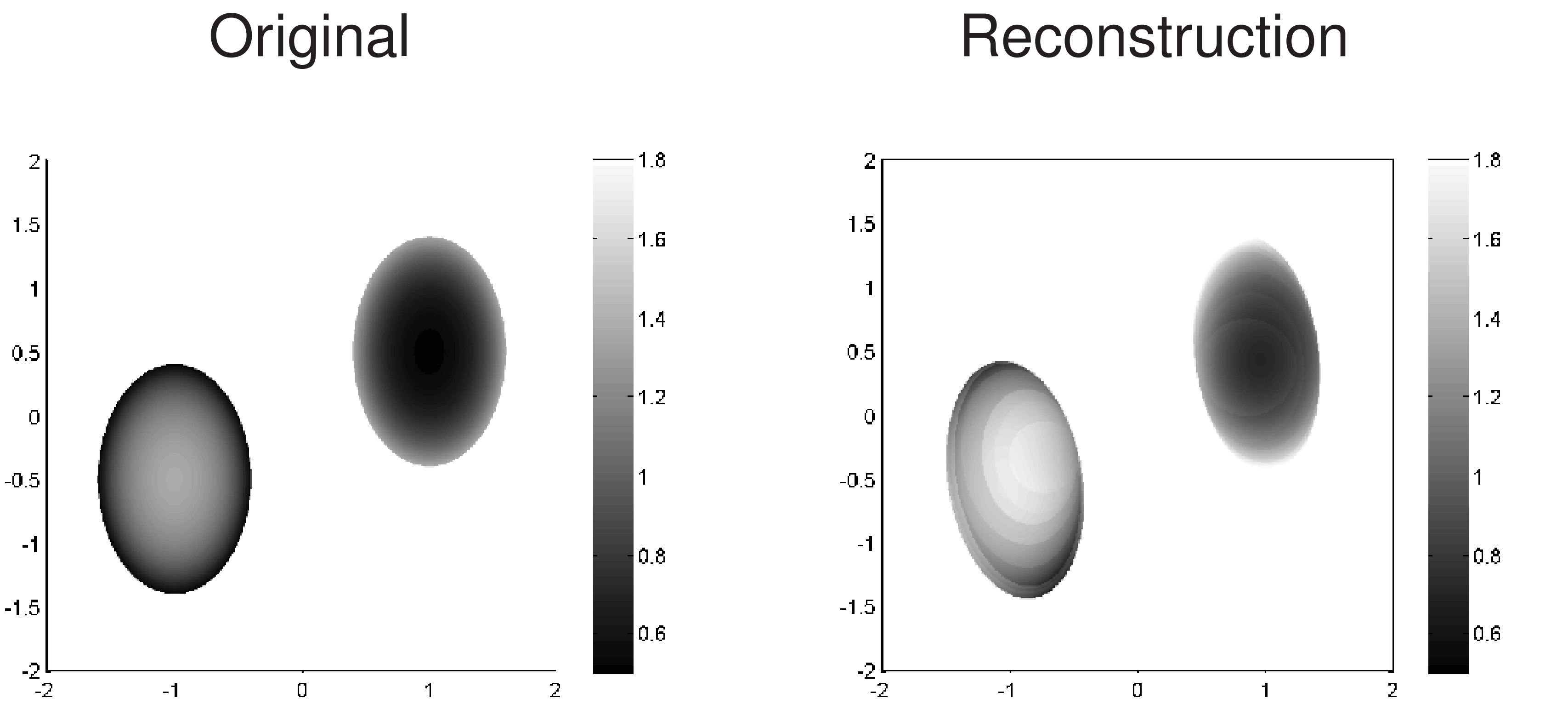
Heterogeneous materials



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Other problems and generalizations

- A non–monotone method
  - Generalization of the concept of the topological derivative
  - Allows to detect annular defects and to remove spurious regions
- Other problems and boundary conditions
  - Dirichlet and Neumann problems:

$$u|_\Gamma = 0 \quad \text{or} \quad \partial_n u|_\Gamma = 0$$

  - General transmission problems:

$$u^+ = u^-, \quad \alpha_+ \partial_n u^+ = \alpha_- \partial_n u^-$$

  - Heterogeneous materials:

$$k = k(\mathbf{x}), \quad \alpha = \alpha(\mathbf{x})$$

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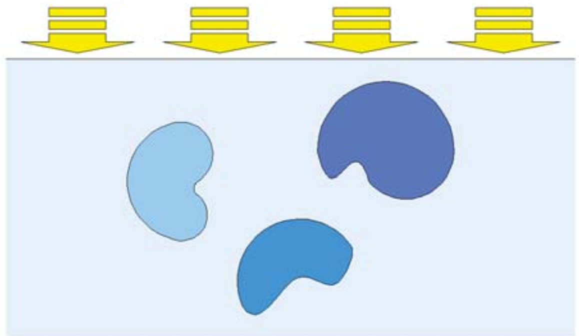
Other problems and generalizations

- Non–steady problems
  - We combine topological derivatives in space with Laplace transforms in time
  - The observation of the system over an interval of time allows for better reconstructions than in the time–harmonic case



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Conclusions

- The topological derivative is a powerful tool to solve inverse problems dealing with shape reconstruction in different areas: acoustics, phototermal problems, elasticity, tomography,...
- The TD gives a good approximation of the number, size and location of the objects buried in a medium
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  - Anisotropic elasticity
  - Electrical impedance tomography

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More information

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